**Lecture No. 9**

Example Application of the FEM using Cardinal Basis Functions

In this example we shall use the Cardinal basis form of the interpolating functions to solve:

$$\frac{dT}{dt} + 2T - 1 = 0$$

initial condition $T = 1$ at $t = 0$. Consider the solution between $0 \leq t \leq 1$

Develop the weighted residual formulation

- The interior error is defined as:

$$\varepsilon_I = \frac{d\hat{T}}{dt} + 2\hat{T} - 1$$

  Note: $\hat{T} = T_{app}$ (new notation)

  Also there is no boundary error since only the i.c.'s and the function are specified.

- We require $\varepsilon_I$ to be orthogonal to a set of weighting functions:

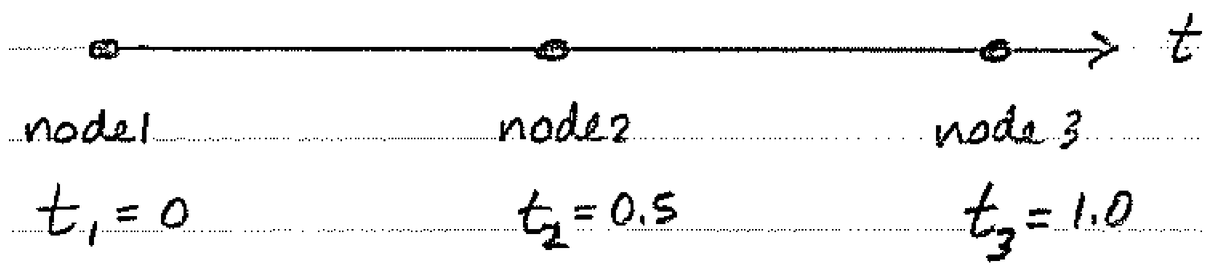
$$\langle \varepsilon_I, w_i \rangle_\Omega = 0, \quad i = 1, N$$

- For Galerkin we have $w_i = \phi_i$.

  Thus the error constraint equation for Galerkin will be:

$$\int_0^1 \left\{ \frac{d\hat{T}}{dt} + 2\hat{T} - 1 \right\} \phi_i dt = 0, \quad i = 1, N$$

- Let us consider the following discretization



Approach 1 - Solving problem with Cardinal functions which satisfy homogenous b.c.'s

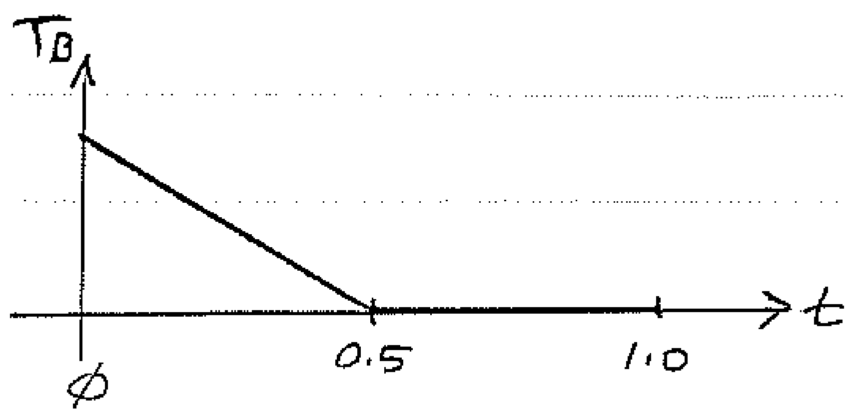
- Let

$$\hat{T} = T_B + \sum_{i=2}^{3} T_i \Phi_i$$

- In order to satisfy b.c. $T(t = 0) = 1$

$$\begin{cases} T_B(t) = \left(\dfrac{0.5 - t}{0.5}\right) & 0 \le t \le 0.5 \\ T_B(t) = 0 & 0.5 \le t \le 1.0 \end{cases}$$

Thus $T_B(t = 0) = 1.0$



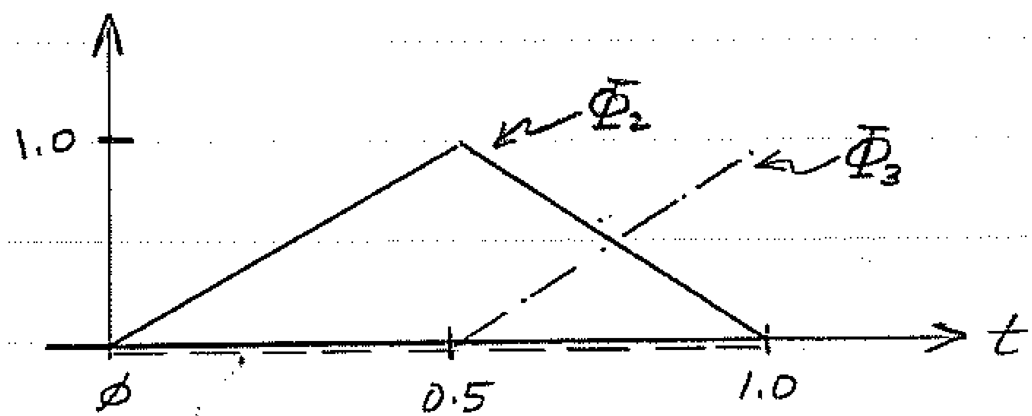
- Also define $\Phi_i \quad i = 2,3$ as the standard Lagrange basis for the given element.

$$\Phi_2 = \frac{t}{0.5} \qquad 0 \le t \le 0.5$$

$$\Phi_2 = \frac{1-t}{0.5} \qquad 0.5 \le t \le 1.0$$

$$\Phi_3 = 0 \qquad 0 \le t \le 0.5$$

$$\Phi_3 = \frac{t-0.5}{0.5} \qquad 0.5 \le t \le 1.0$$



- Thus $\hat{T} = T_B + \sum_{i=2}^{3} T_i\, \Phi_i$ satisfies functional continuity and boundary/initial condition

$$\therefore \hat{T} = T_B + T_2\Phi_2 + T_3\Phi_3$$

- Substituting into the weighted residual statement:

$$\int_0^1 \left\{\frac{d}{dt}(T_B + T_2\Phi_2 + T_3\Phi_3) + 2(T_B + T_2\Phi_2 + T_3\Phi_3) - 1\right\}\Phi_i dt = 0 \quad i = 2,3$$

- For $i = 2 \quad \Phi_2$

$$\int_0^1 \left\{\frac{d}{dt}(T_B + T_2\Phi_2 + T_3\Phi_3) + 2(T_B + T_2\Phi_2 + T_3\Phi_3) - 1\right\}\Phi_2 dt = 0$$

$$\Rightarrow$$

$$\int_0^1 \left\{\left(\frac{dT_B}{dt} + 2T_B\right)\Phi_2 + \left(\frac{d\Phi_2}{dt} + 2\Phi_2\right)\Phi_2 T_2 + \left(\frac{d\Phi_3}{dt} + 2\Phi_3\right)\Phi_2 T_3 - \Phi_2\right\} dt = 0$$

Since $T_B$ is only nonzero within $\left[0, \frac{1}{2}\right]$

Since $\Phi_3$ is only nonzero within $\left[\frac{1}{2}, 1\right]$

$$\boxed{\int_0^{\frac{1}{2}} \left(\frac{dT_B}{dt} + 2T_B\right)\Phi_2 dt + \left\{\int_0^1 \left(\frac{d\Phi_2}{dt} + 2\Phi_2\right)\Phi_2 dt\right\} T_2 + \left\{\int_{\frac{1}{2}}^1 \left(\frac{d\Phi_3}{dt} + 2\Phi_3\right)\Phi_2 dt\right\} T_3 = \int_0^1 \Phi_2 dt}$$

- For $i = 3$

$$\int_0^1 \left\{\frac{d}{dt}(T_B + T_2\Phi_2 + T_3\Phi_3) + 2(T_B + T_2\Phi_2 + T_3\Phi_3) - 1\right\}\Phi_3\, dt = 0$$

$\Rightarrow$

$$\int_0^1 \left\{\left(\frac{dT_B}{dt} + 2T_B\right)\Phi_3 + \left(\frac{d\Phi_2}{dt} + 2\Phi_2\right)\Phi_3 T_2 + \left(\frac{d\Phi_3}{dt} + 2\Phi_3\right)\Phi_3 T_3 - \Phi_3\right\}dt = 0$$

Since $T_B$ is only nonzero within $\left[0, \frac{1}{2}\right]$

Since $\Phi_3$ is only nonzero within $\left[\frac{1}{2}, 1\right]$

$$\boxed{\left\{\int_{\frac{1}{2}}^1 \left(\frac{d\Phi_2}{dt} + 2\Phi_2\right)\Phi_3\, dt\right\}T_2 + \int_{\frac{1}{2}}^1 \left(\frac{d\Phi_3}{dt} + 2\Phi_3 dt\right)\Bigg\}T_3 = \int_{\frac{1}{2}}^1 \Phi_3 dt}$$

- Working out all the integrals:

$$i = 2 \qquad \frac{1}{2}\left[-\frac{2}{3} + \frac{4}{3}T_2 + \frac{4}{3}T_3\right] = \frac{1}{2}(1)$$

$$i = 3 \qquad \frac{1}{2}\left[0 - \frac{2}{3}T_2 + \frac{5}{3}T_3\right] = \frac{1}{2}\left(\frac{1}{2}\right)$$

This leads to the following 2x2 system

$$\boxed{\frac{1}{2}\begin{bmatrix} \frac{4}{3} & \frac{4}{3} \\ -\frac{2}{3} & \frac{5}{3} \end{bmatrix}\begin{bmatrix} T_2 \\ T_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 + \frac{2}{3} \\ \frac{1}{2} \end{bmatrix}}$$

We can now solve this system and for $T_2$ and $T_3$

- <u>Approach 2</u>

Let's now no longer define a function $T_B$ but only consider basis functions in formulating $\hat{T}$

$$\hat{T} = \sum_{j=1}^{3} T_j \Phi_j(t)$$

$T_j$ = global value of solution at all three nodes

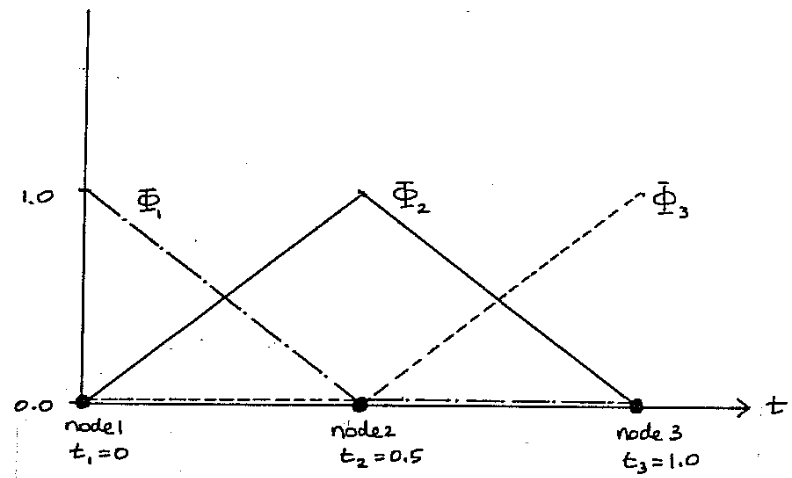
$\Phi_j$ = Cardinal basis function for local Lagrange type linear interpolation

We will actually implement the *i.c./b.c.* in the system of simultaneous equations once it has been formed

- Let's define our elements over the domain:

  Use 2 elements between $0 \leq t \leq 1$ and use linear chapeau interpolating functions

  This defines the following 3 Cardinal functions and 3 nodes.




These three functions are defined as:

$$\Phi_i = \begin{cases} \frac{t-t_{i-1}}{t_i-t_{i-1}} & t_{i-1} \leq t \leq t_i \\ \frac{t_{i+1}-1}{t_{i+1}-1} & t_i \leq t \leq t_{i+1} \end{cases} \text{ and zero elsewhere}$$

- Thus we define the approximating sequence as:

$$T \cong \hat{T} = \sum_{j=1}^{3} T_j \Phi_j(t)$$

There are 3 unknown coefficients $T_j$ at the nodes. The way we defined our functions, $\Phi_j$, these coefficients equal the function value at the nodes.

- Substituting $\hat{T}$ into the weighted residual statement:

$$\int_0^1 \left\{ \frac{d}{dt}\left( \sum_{j=1}^{3} T_j \Phi_j(t) \right) + 2 \left( \sum_{j=1}^{3} T_j \Phi_j(t) \right) - 1 \right\} \Phi_i dt = 0 \quad i = 1{,}3$$

$$\Rightarrow$$

$$\int_0^1 \left\{ \sum_{j=1}^{3} T_j \left( \frac{d\Phi_j}{dt} + 2\, \Phi_j \right) - 1 \right\} \Phi_i dt = 0 \quad i = 1{,}3$$

This generates a system of equations of rank 3

- Let $i = 1$

$$\int_0^1 \left\{T_1\left(\frac{d\Phi_1}{dt} + 2\Phi_1\right) + T_2\left(\frac{d\Phi_2}{dt} + 2\Phi_2\right) + T_3\left(\frac{d\Phi_3}{dt} + 2\Phi_3\right) - 1\right\}\Phi_1 dt = 0$$

We note that $\Phi_1 = 0$ for $t > \frac{1}{2}$ and also $\Phi_3 = 0$ for $t \leq \frac{1}{2}$ (thus the third term drops entirely):

$$\left[\int_0^{\frac{1}{2}} \left(\frac{d\Phi_1}{dt} + 2\Phi_1\right)\Phi_1 dt\right] T_1 + \left[\int_0^{\frac{1}{2}} \left(\frac{d\Phi_2}{dt} + 2\Phi_2\right)\Phi_1 dt\right] T_2 = \left[\int_0^{\frac{1}{2}} 1\Phi_1 dt\right]$$

Following similar procedures for $i$ = 2 and 3 leads to the following system of equations:

$$\begin{bmatrix} \int_0^{1/2} \left(\frac{d\Phi_1}{dt}\Phi_1 + 2\Phi_1\Phi_1\right)dt & \int_0^{1/2} \left(\frac{d\Phi_2}{dt}\Phi_1 + 2\Phi_2\Phi_1\right)dt & 0 \\ \int_0^{1/2} \left(\frac{d\Phi_1}{dt}\Phi_2 + 2\Phi_1\Phi_2\right)dt & \int_0^{1} \left(\frac{d\Phi_2}{dt}\Phi_2 + 2\Phi_2\Phi_2\right)dt & \int_{1/2}^{1} \left(\frac{d\Phi_3}{dt}\Phi_2 + 2\Phi_3\Phi_2\right)dt \\ 0 & \int_{1/2}^{1} \left(\frac{d\Phi_2}{dt}\Phi_3 + 2\Phi_2\Phi_3\right)dt & \int_{1/2}^{1} \left(\frac{d\Phi_3}{dt}\Phi_3 + \Phi_3\Phi_3\right)dt \end{bmatrix}$$

$$\cdot \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} \int_0^{1/2} 1 \cdot \Phi_1 dt \\ \int_0^{1} 1 \cdot \Phi_2 dt \\ \int_{1/2}^{1} 1 \cdot \Phi_3 dt \end{bmatrix}$$

The matrix will not be symmetrical due to the nature of the first derivative operator (since $L$ is not self adjoint). Evaluating the required integrals leads to:

$$\frac{1}{2}\begin{bmatrix} -\frac{1}{3} & \frac{4}{3} & 0 \\ -\frac{2}{3} & \frac{4}{3} & \frac{4}{3} \\ 0 & -\frac{2}{3} & \frac{5}{3} \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \end{bmatrix}$$

Equation 1 is associated with node 1

Equation 2 is associated with node 2

Equation 3 is associated with node 3

- However our i.c. states that

$$T(t=0) = 1 \Rightarrow T_1 = 1$$

***We now have 3 unknowns and 4 equations. We must eliminate 1 equation. The logical and best choice is to eliminate the equation associated with $T_1$, the first equation. This leads to the following system of equations.***

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & \frac{4}{3} & \frac{4}{3} \\ 0 & -\frac{2}{3} & \frac{5}{3} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \frac{1}{2} \end{bmatrix} \xrightarrow{\text{can contract system}} \begin{bmatrix} \frac{4}{3} & \frac{4}{3} \\ -\frac{2}{3} & \frac{5}{3} \end{bmatrix} \begin{bmatrix} T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 1 + \frac{2}{3} \\ \frac{1}{2} \end{bmatrix}$$

This system of equations is identical to those found in approach 1!! Thus essential b.c.'s can be incorporated by an equation substitution procedure.

- Thus it's very easy to treat function specified boundary conditions such that they are exactly satisfied.
- Solving the system of equation:

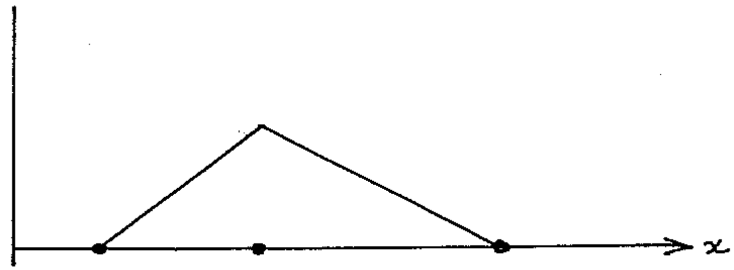
$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 1.000 \\ 0.678 \\ 0.571 \end{bmatrix}$$

- Problems regarding programming convenience when using Cardinal basis occur when:

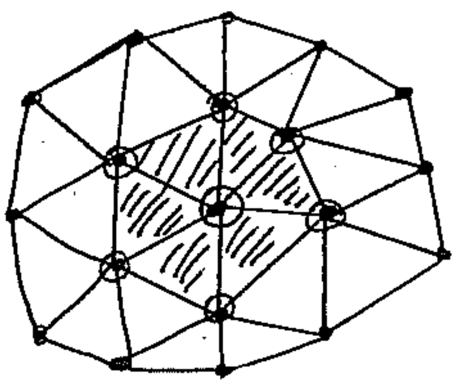
1. Using higher order basis, the number of nodes over which they are defined increases and varies depending on the functions. We must determine where which function is identically defined as zero. This is quite cumbersome.
2. Working in higher spatial dimensions, the nodal and element connectivity becomes much more complex.

1-Dimension

Each element has 2 adjacent elements (and chapeau functions are defined over only 2 elements and 3 nodes).




2-Dimensions



Cardinal functions for 1 node extends over 6 elements and 7 total nodes.

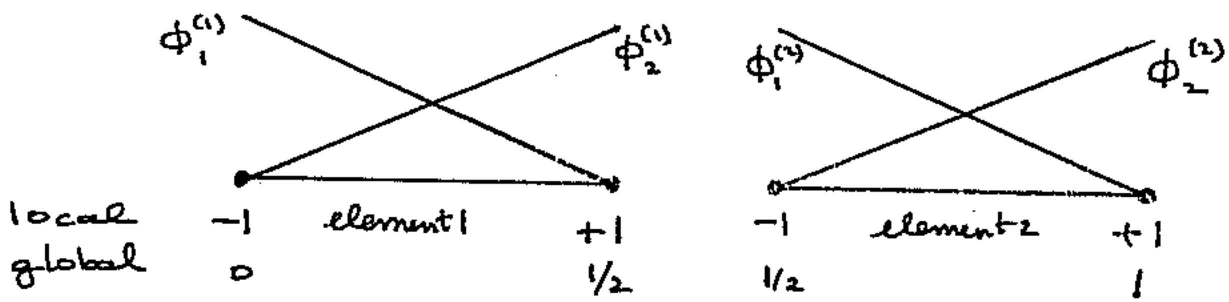
They are zero elsewhere

- Thus integration in the global domain becomes much more difficult.
- To prevent the programming complexities associated with cardinal basis functions, we shall consider the elements separately (i.e. not considering the other elements) and work with localized (or elemental and not Cardinal) functions.

- The elements are then joined together such that the necessary continuity conditions are satisfied. Finally global b.c.'s are enforced.
- Recall that Cardinal basis were formed by adding individual localized element functions together and taking into account continuity. However working on an elemental basis we can really do the same thing.

Approach 3

- Use a localized formulation and implement $C_0$ functional continuity constrains in the "global matrix assembly" process
- This is FEM using Localized Basis Functions
- We now consider our two elements separately. Each has 2 localized interpolating functions associated with it.
- The 2 functions $\phi_i^{(n)}$ are defined identically for all elements (at least on a local coordinate system).



- Localized functions are zero everywhere except within a given element. For all elements *n*. (Note *n* will represent the element number)

$$\phi_1^{(n)} = \frac{1}{2}(1-\xi)$$

$$\phi_2^{(n)} = \frac{1}{2}(1+\xi)$$

and

$$\hat{T}^{(n)} = \sum_{i=1}^{2} T_i^{(n)} \phi_i^{(n)}$$

- Thus for each element we have 2 unknown nodal coefficients, $T_1^{(n)}, T_2^{(n)}$. The unknowns are localized (i.e., one set for each element). We'll take care of functional continuity later. The weighted residual statement now becomes:

$$\sum_{el} \int_{\Omega_e} \left\{ \frac{d\hat{T}^{(n)}}{dt} + 2\hat{T}^{(n)} - 1 \right\} \phi_i^{(n)} dt = 0 \quad i = 1,2$$

- We must:
    - Consider each element separately and perform the weighting associated with the element functions.
    - Sum up over the domain and take into account functional continuity.

- Element 1 ($n$ = 1)

$$\int_0^{\frac{1}{2}} \left\{\frac{d\hat{T}^{(1)}}{dt} + 2\hat{T}^{(1)} - 1\right\} \phi_i^{(1)} dt = 0 \quad i = 1,2$$

However we want to work in localized coordinates $\xi$. We found that the appropriate transformation for derivatives was:

$$\frac{d\hat{T}^{(n)}(t)}{dt} = \frac{d\hat{T}^{(n)}(\xi)}{d\xi}\frac{d\xi}{dt} = \frac{2}{\Delta t^{(n)}}\frac{d\hat{T}^{(n)}(\xi)}{d\xi}$$

where $\Delta t^{(n)}$ = global length of element $n$

Furthermore:

$$dt = \frac{\Delta t^{(n)}}{2} d\xi$$

and the limits change to:

$$\int_{0}^{\frac{1}{2}} \rightarrow \int_{-1}^{+1}$$

Thus in local coordinates for element $n = 1$:

$$\frac{\Delta t^{(1)}}{2} \int_{-1}^{+1} \left\{ \frac{2}{\Delta t^{(1)}} \frac{d\hat{T}^{(1)}}{d\xi} + 2\hat{T}^{(1)} - 1 \right\} \phi_i^{(1)}(\xi)\, d\xi = 0 \quad \mathrm{i} = 1{,}2$$

However we recall that

$$\hat{T}^{(1)} = T_1^{(1)} \phi_1^{(1)} + T_2^{(1)} \phi_2^{(1)}$$

Substituting we have:

$$\frac{\Delta t^{(1)}}{2} \int_{-1}^{+1} \left\{ \frac{2}{\Delta t^{(1)}} \left( T_1^{(1)} \frac{d\phi_1^{(1)}}{d\xi} + T_2^{(1)} \frac{d\phi_2^{(1)}}{d\xi} \right) + 2\left( T_1^{(1)} \phi_1^{(1)} + T_2^{(1)} \phi_2^{(1)} \right) - 1 \right\} \phi_i^{(1)} d\xi = 0$$

$$\mathrm{i} = 1{,}2$$

This leads to a 2 x 2 system of equations:

$$\frac{1}{2} \begin{bmatrix} -\frac{1}{3} & \frac{4}{3} \\ -\frac{2}{3} & \frac{5}{3} \end{bmatrix} \begin{bmatrix} T_1^{(1)} \\ T_2^{(1)} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

- Element 2 ($n$=2)

  Following the same procedure, we find that for element 2:

$$\frac{1}{2}\begin{bmatrix} -\frac{1}{3} & \frac{4}{3} \\ -\frac{2}{3} & \frac{5}{3} \end{bmatrix}\begin{bmatrix} T_1^{(2)} \\ T_2^{(2)} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

- Now we must perform the global summation($\sum_{el}$ ) in addition to accounting for inter-element functional continuity. First we relate local unknowns to global unknowns.

| local | global |
|---|---|
| $T_1^{(1)}$ | $T_1$ |
| $T_2^{(1)}$ | $T_2$ |
| $T_1^{(2)}$ | $T_2$ |
| $T_2^{(2)}$ | $T_3$ |

Note that $T_2^{(1)} = T_1^{(2)} = T_2$ represent the same functional value at the same node.

- Adding each local entry into the appropriate global location we have:

$$\frac{1}{2}\begin{bmatrix} -\frac{1}{3} & \frac{4}{3} & \\ -\frac{2}{3} & \left(\frac{5}{3}-\frac{1}{3}\right) & \frac{4}{3} \\ & -\frac{2}{3} & \frac{5}{3} \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2}+\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

This is the same matrix as was found with the cardinal formulation, only it's <u>much</u> easier to assemble it (just loop through all the elements in the grid). Now we set b.c.'s and solve

<u>Notes on the integrations involved</u>

- Typically we can develop general algebraic expressions for each elemental matrix and elemental load vector for each element which depend only on element geometry and material properties.
- As complexity and/or interpolation order increases it may be simpler to use numerical integrators.
- Also we like our programs to allow for the use of any order interpolation. It's much easier to use numerical integrators to accomplish this.

## Summary Notes on Global vs Local Expansions

In solving $\frac{d\hat{T}}{dt} + 2\hat{T} - 1 = 0 \qquad T(t=0) = T^*$ (specified i.c.)

we have so far applied 3 approaches

Approach 1: We developed a global expansion (using Cardinal functions) and properly define a boundary function prior to formulating the weighted residual equation

$$\hat{T} = T_B + \sum_{j=2}^{3} T_j \Phi_j$$

$$T_B = T^* \Phi_1$$

Thus $T_B$ satisfied the i.c. and $\Phi_j \quad j = 1,2$ satisfy the homogeneous b.c.'s. Now plug through weighted residual formulation.

Approach 2: Develop global expansion which does not account for i.c.'s. Plug through w.r. formulation, then form matrix system, then at the very end enforce the i.c.'s.

$$\hat{T} = \sum_{j=1}^{3} T_j \, \Phi_j$$

$$\Rightarrow$$

$$\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

Then define $T_B = T_1\Phi_1 = T^*\Phi$ by enforcing $T_1 = T^*$

Thus we've really reduced the number of unknowns to only two ⇒ therefore we must reduce the number of constraint equations !!!

We must eliminate the first orthogonality constraint equation by in arrears setting $W_1 = 0 \Rightarrow$ thereby eliminating 1st equation and one of the constraint equations!

Now we have the correct number of constraint equations and unknowns

∴ Keep $\hat{T} = \sum_{j=1}^{3} T_j\Phi_j$ and we in arrears set $\hat{T} = T^*\Phi_1 + \sum_{j=2}^{3} T_j\Phi_j$

Approach 3: In this approach we just formulated 2 localized problems and implemented seemingly separate w.r. formulation. Then we assembled them into a global system

In element 1

$$\hat{T}^1 = T_1^1 \phi_1^1 + T_2^1 \phi_2^1$$

$$\langle \varepsilon_I, \phi_j^1 \rangle_{\Omega_{el1}} = 0 \qquad j = 1,2$$

$$\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} T_1^1 \\ T_2^1 \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \end{bmatrix}$$

In element 2

$$\hat{T}^2 = T_1^2 \phi_1^2 + T_2^2 \phi_2^2$$

$$\langle \varepsilon_I, \phi_j^2 \rangle_{\Omega_{el2}} = 0 \qquad j = 1,2$$

$$\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} T_1^2 \\ T_2^2 \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \end{bmatrix}$$

Now we can globalize the variables (i.e. enforcing functional continuity)

$$T_1^1 \rightarrow T_1$$
$$T_2^1 \rightarrow T_2$$
$$T_1^2 \rightarrow T_2$$
$$T_2^2 \rightarrow T_3$$

Finally we can sum into a global system

Approach 4: Let's work out approach 3 in a little more detail

First we define the approximation as the sum over the element of local expansion

$$\hat{T} = \sum_{k=1}^{M} \sum_{i=1}^{2} T_i^k \phi_i^k$$

$$\hat{T} = \sum_{k=1}^{M} \left( T_1^k \phi_1^k + T_2^k \phi_2^k \right)$$

Note $M$ = # elements and we assume linear bases locally over each element $k$

Let's worry about enforcing i.c.'s (or b.c.'s) at the very end of the problem (like we did in #2 and #3)

Define weighted residual statement

$$\varepsilon_I = \frac{d\hat{T}}{dt} + 2\hat{T} - 1$$

Since we currently have 2M unknown coef.'s, we need to enforce 2M constraints

$$\langle \varepsilon_I, \phi_j^k \rangle_\Omega = 0 \qquad j = 1,2; \; k = 1, M$$

$$\Rightarrow \int_\Omega \left\{ \frac{d}{dt} \sum_{k=1}^{M} \left( T_1^k \phi_1^k + T_2^k \phi_2^k \right) + 2 \sum_{k=1}^{M} \left( T_1^k \phi_1^k + T_2^k \phi_2^k \right) - 1 \right\} \phi_j^k dt = 0 \quad j = 1,2 \quad k = 1, M$$

Since for the ivp we've built in the time dependence into the bases

$$\int_{\Omega} \left\{ \sum_{k=1}^{M} \left( T_1^k \frac{d\phi_1^k}{dt} + T_2^k \frac{d\phi_2^k}{dt} \right) + 2 \sum_{k=1}^{M} \left( T_1^k \phi_1^k + T_2^k \phi_2^k \right) - 1 \right\} \phi_j^k \, dt = 0 \quad j = 1,2 \quad k = 1, M$$

Now we note that we <u>must</u> take care of functional continuity constraints. Thus we must enforce

$$T_2^1 \to T_1^2$$

$$T_2^2 \to T_1^3$$

$$T_2^3 \to T_1^4$$

<u>However</u> as we enforce each inter-element constraints, we must eliminate/modify a constraint equation. We also note that since the weighting functions must match the bases, thus we <u>must</u> actually consolidate select weighting functions.

Thus on inter-element boundaries

For elements 1/2

$$\left.\begin{matrix} \langle \varepsilon_I, \phi_2^1 \rangle_{\Omega} = 0 \\ \langle \varepsilon_I, \phi_1^2 \rangle_{\Omega} = 0 \end{matrix}\right\} \Rightarrow$$

$$\langle \varepsilon_I, \phi_2^1 + \phi_1^2 \rangle_{\Omega} = 0 \Rightarrow$$

$$\langle \varepsilon_I, \phi_2^1 \rangle_{\Omega_1} + \langle \varepsilon_I, \phi_1^2 \rangle_{\Omega_1} = 0$$

For elements 2/3

$$\left.\begin{aligned}\langle \varepsilon_I, \phi_2^2 \rangle_\Omega = 0\\ \langle \varepsilon_I, \phi_1^3 \rangle_\Omega = 0\end{aligned}\right\} \Rightarrow$$

$$\langle \varepsilon_I, \phi_2^2 + \phi_2^3 \rangle_\Omega = 0 \Rightarrow$$

$$\langle \varepsilon_I, \phi_2^1 \rangle_{\Omega_2} + \langle \varepsilon_I, \phi_1^3 \rangle_{\Omega_3} = 0$$

etc.

The first and last nodes stay the same

$$\langle \varepsilon_I, \phi_1^1 \rangle_{\Omega_1} = 0$$

$$\langle \varepsilon_I, \phi_1^M \rangle_{\Omega_M} = 0$$

(Note that I’ve accounted for where the bases are equal to zero)

This then explains why we add equations at each interface node together.